MOTION CONTROL OF AN UNDERACTUATED WHEELED MOBILE ROBOT: A KINEMATIC INPUT-OUTPUT LINEARIZATION APPROACH

Werther Alexandre de Oliveira Serralheiro
Newton Maruyama
Eduardo Aoun Tannuri
Departamento de Engenharia Mecatônica e de Sistemas Mecânicos
Escola Politécnica da Universidade de São Paulo
Av. Prof. Mello Moraes, 2231, São Paulo, Brasil
e-mails: {werther,maruyama,eduat}@usp.br

Abstract. This work addresses a trajectory tracking controller for the kinematics of an underactuated wheeled mobile robot, in environments without obstacles, using input-output linearization and the Follow the Carrot algorithm. The use of input-output linearization allows the reduction of error dynamics into a linear order system. In this case, the controller is expressed as a $2 \times 2$ diagonal gain matrix. Different types of trajectories are devised in order to analyse the feasibility of the method. A specific trajectory consisting of a mix of smooth and sharp curvature sections is utilized in order to investigate the sensitivity of the control system in relation to gain variations. Some novelty is claimed by authors in relation to the use of kinematic input-output linearization combined to the Follow the Carrot algorithm.

Keywords: wheeled mobile robots, trajectory tracking, motion control, nonlinear control system, input-output linearization.

1. Introduction

Over the last few decades the research on motion control of mobile robots has been rapidly increasing. Under motion control it is possible to identify three basic problems:

- Point stabilization: the vehicle is required to stabilize at a given pose (position and orientation);
- Path following: the vehicle is required to follow a given geometric path;
- Trajectory tracking: the vehicle is required to track a trajectory, i.e., a geometric path with temporal properties.

Since there is no temporal specification in the path following problem, the performance is expressed by the minimum distance $d$ from the vehicle to the geometric path. In the trajectory tracking problem the control system performance is expressed by the distance $e$ from the vehicle to a reference point as shown on Figure 1.

Figure 1. Path following versus Trajectory tracking.

Several path following and trajectory tracking control algorithms have been described in the mobile robotics literature (Svec et al., 2014). Classical methods, such as the Follow the Carrot (Barton, 2001) and the Pure Pursuit (Coulter, 1992), use robot position information to compute steering commands in order to follow a predefined geometric path. Variations of these algorithms are also found in the literature, see for example, (Hogg et al., 2002) and (Sujit et al., 2013). These algorithms are known to have poor performance in corners since they do not take into account the actual curvature of the path.
More complex algorithms have been developed recently. The Follow the Past (Hellstrom and Ringdahl, 2006) uses recorded steering commands information to overcome the problem with sharp trajectory tracking found in the classical methods; the Vector Pursuit (Wit et al., 2004; Yeu et al., 2006) is a geometric path following method based on the screw theory; the Valued-based controller (Bohren et al., 2008) integrates the dynamics of the vehicle model in order to predict optimal steering commands; a robust Model Reference Adaptive Controller has also been studied for mobile robots with uncertainties in the dynamical model (Aneesh, 2012).

Encarnação and Pascoal (2002) has introduced a combined trajectory tracking and path following control approach for wheeled robots. Other examples are (Aguiar and Hespanha, 2007; Xiang et al., 2011; Alessandretti et al., 2013).

This work addresses a trajectory tracking controller for the kinematics of an underactuated wheeled mobile robot, using feedback linearization and a typical motion control scheme: the Follow the Carrot algorithm. Although feedback linearization has been proposed in motion control of WMR (Kim and Oh, 1999; Oriolo et al., 2002; Chwa, 2010; Akbati and Cansever, 2013), this paper claims novelty while combining kinematics feedback linearization and the Follow the Carrot algorithm.

The paper is organized as follows. In Section 2, a kinematic model formulation of a WMR is introduced. A control system scheme using feedback linearization and the Follow the Carrot approach is presented in Section 3. In Section 4, some simulation results are presented and discussed. Finally in Section 5 some conclusions about the feasibility of the proposed method are drawn.

2. The kinematic model

A differential wheeled mobile robot (WMR) is made up of a rigid frame and equipped with nondeformable driven wheels, as illustrated by the schematic diagram on Figure 2, where \( b \) is the vehicle width, \( l \) is the vehicle length and \( r \) the vehicle wheels radii.

![Figure 2. The differential WMR and the coordinate frames.](image)

It is assumed that the robot is moving on a horizontal plane and an arbitrary inertial frame \( \{O\} = \{X_O, Y_O\} \) is fixed in the motion plane. The robot coordinate frame \( \{X_R, Y_R\} \) is attached to the robot chassis on the reference point given by \( R \), which is positioned in the middle of the shaft. The position coordinates of \( R \) in relation to the inertial frame is given by \( x \) and \( y \), and the rotation angle between both coordinates frames is given by \( \theta \). The WMR pose is then completely defined by the following vector:

\[
\xi_O(t) = [x(t) \ y(t) \ \theta(t)]^T. \tag{1}
\]

As both wheels are individually controlled it results that the angular wheel velocities \( \omega_r(t) \) e \( \omega_l(t) \) are independent. Their relationship with the robot translation velocity \( v(t) \) and with the angular velocity \( \dot{\theta} = \omega(t) \) can be written by the following equations (Nørgaard et al., 2000):
And the error derivatives become:

\[ v(t) = \frac{r(\omega_r(t) + \omega_l(t))}{2} \]
\[ \omega(t) = \frac{r(\omega_r(t) - \omega_l(t))}{b} \]

(2)

The velocities of \( R \) in the inertial coordinate frame \( \{ O \} \) are given by \( \dot{x}(t) = v(t)\cos\theta(t) \) and \( \dot{y}(t) = v(t)\sin\theta(t) \). Then, the pose time derivative (kinematic model) can be written to yield:

\[ \dot{\xi}_O = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} r\cos\theta(t)/2 & r\cos\theta(t)/2 & 0 \\ r\sin\theta(t)/2 & r\sin\theta(t)/2 & 0 \\ r/b & -r/b & 0 \end{bmatrix} \begin{bmatrix} \omega_r(t) \\ \omega_l(t) \end{bmatrix} \]

(3)

Note that the unicycle is underactuated, since \( \dot{\xi}_O \in \mathbb{R}^3 \) and the control signal \( u = [\omega_r(t), \omega_l(t)]^T \in \mathbb{R}^2 \).

3. The control system

3.1 Feedback Linearization

The feedback linearization approach (Slotine and Li, 1991) is based on the cancelling of system nonlinearities while imposing desired linear dynamics. The central idea is to algebraically transform nonlinear system dynamics into fully or partially linear ones, so that linear control techniques might be applied.

Let a system described by the companion form \( \dot{x} = f_1(x) + f_2(x)u \), where \( u \in \mathbb{R}^p \) is the control input, \( x \in \mathbb{R}^n \) is the state vector and \( f_1(x) \) and \( f_2(x) \) are nonlinear function of states. Using the control input given by:

\[ u = (f_2)^{-1}[v - f_1], \]

(4)

and if \( f_2 \) is not singular the nonlinearities can be cancelled and an input-output relation \( \dot{x} = \nu \) is obtained. The control law \( \nu \) must be chosen such that the internal dynamics becomes stable. This linearization approach has been efficiently applied for WMR motion control, see for example (D’Andrea-Novel et al., 1992; Kim and Oh, 1999; Oriolo et al., 2002; Chwa, 2010).

3.2 The Follow the Carrot scheme

The WMR trajectory tracking consists in calculating the robot velocity and heading in order to force the robot to follow a predefined pose set. The Follow the Carrot (Barton, 2001; Sujit et al., 2013) trajectory tracking approach originates from the idea of holding a carrot in front of a horse to force the animal to move in desired direction.

\( C_p = (x_{ref}, y_{ref}) \in \{ O \} \) is the coordinates of the Carrot Point moving in a predefined time parametrized and obstacle free geometric path\(^1\). The vector \( c_{vp} \) represents the Carrot Point velocity in the plane. A virtual segment line is drawn from the center \( R = (x, y) \in \{ O \} \) of the WMR to the Carrot Point \( C_p \). Also a distance gap \( \rho \) within the same virtual segment line is considered to avoid singularity problems.

Based on these considerations, two errors are here defined the linear error \( e_t \) from the WMR to the distance gap \( \rho \); and the heading error \( (e_\theta) \) between the WMR direction and the Carrot Point \( C_p \), as illustrated by Figure 3.

The components of the linear error in the \( X_O \) and the \( Y_O \) directions are respectively given by \( e_x = (x_{ref} - x) \) and \( e_y = (y_{ref} - y) \). Then, the errors \( e_t \) and \( e_\theta \) can be written as:

\[ e_t = \sqrt{e_x^2 + e_y^2} - \rho, \]
\[ e_\theta = \tan^{-1}\left( \frac{e_y}{e_x} \right) - \theta. \]

(5)

And the error derivatives become:

\[ \dot{e}_t = e_x(\dot{x}_{ref} - \dot{x}) + e_y(\dot{y}_{ref} - \dot{y}), \]
\[ \dot{e}_\theta = e_x(\dot{y}_{ref} - \dot{y}) - e_y(\dot{x}_{ref} - \dot{x}) \]
\[ - \dot{\theta}. \]

(6)

\(^1\)the explicit functional dependency on time \( t \) is neglected here in order to simplify the equations.
Substituting the kinematic unicycle model given by the Equation 3 into the Equation 6 and after some algebraic manipulations yields:

\[
\begin{bmatrix}
\dot{e}_l \\
\dot{e}_\theta
\end{bmatrix} = f_{1R} \begin{bmatrix}
\dot{x}_{ref} \\
\dot{y}_{ref}
\end{bmatrix} + f_2 \begin{bmatrix}
\omega_r \\
\omega_l
\end{bmatrix},
\]  

(7)

where the nonlinear functions are given by:

\[
f_{1R} = \begin{bmatrix}
+ \left( \frac{e_x}{e_l} \right) + \left( \frac{e_y}{e_l} \right) \\
- \left( \frac{e_x}{e_l^2} \right) + \left( \frac{e_y}{e_l^2} \right)
\end{bmatrix},
\]

\[
f_2 = \begin{bmatrix}
- \mathcal{A} & - \mathcal{B} \\
- \mathcal{B} & - \mathcal{A}
\end{bmatrix}.
\]

With the terms inside the \( f_2 \) matrix given by:

\[
\mathcal{A} = \frac{r}{2e_l} \left( e_x \cos \theta + e_y \sin \theta \right),
\]

(9)

\[
\mathcal{B} = \frac{r}{2e_l} \left( e_x \sin \theta - e_y \cos \theta \right).
\]

The control output becomes:

\[
\begin{bmatrix}
\omega_r \\
\omega_l
\end{bmatrix} = f_2^{-1} \cdot \left( K \begin{bmatrix}
e_l \\
e_\theta
\end{bmatrix} - f_{1R} \begin{bmatrix}
\dot{x}_{ref} \\
\dot{y}_{ref}
\end{bmatrix} \right),
\]

(10)

where \( K = \begin{bmatrix}
K_l & 0 \\
0 & K_\theta
\end{bmatrix} \) is a diagonal matrix of gains. The dynamics of errors become:

\[
\begin{bmatrix}
\dot{e}_l \\
\dot{e}_\theta
\end{bmatrix} = K \begin{bmatrix}
e_l \\
e_\theta
\end{bmatrix}
\]

(11)

The errors \( e_l \) and \( e_\theta \) converge to zero if and only if the eigenvalues of \( K \) are negative.

### 3.3 Computational implementation

The kinematic model Equation 3 is implemented as an open loop system, with the geometric parameters vehicle chosen as width \( b = 0.3m \) and wheel radius \( r = 0.1m \). A predefined time parametrized trajectory defines \([x_{ref}(t) \ y_{ref}(t)]^T\), with a time discretization of \( \delta t = 0.05s \).

Each one the errors \( e_x, e_y, e_l \) and \( e_\theta \) are calculated, as well as the time derivatives of the reference trajectory \([\dot{x}_{ref}(t) \ \dot{y}_{ref}(t)]^T\). A distance gap \( \rho = 0.05m \) is chosen. Then, the closed loop system is calculated utilizing the Equation 10.

The complete closed loop control system is illustrated as a block diagram in Figure 4.
4. Experimental results

This section illustrates some experimental results. Initially, with the gain matrix tuned as $K = \begin{pmatrix} -0.1 & 0 \\ 0 & -2.0 \end{pmatrix}$, some trajectories are tested as illustrated in Figure 5: a straight line and a circular pattern, both with constant velocity; a sinusoidal pattern with velocity component in the $y$ direction constant and a spiral trajectory with exponential velocity.

In all performed simulations, the WMR motion reaches predefined geometrical paths and tracks the reference which is defined by the movement of the Carrot Point $C_P$. Furthermore, a specific trajectory, with a mix of smooth and sharp curvature sections, is chosen in order to foster discussions about the effectiveness of this approach. The trajectory tracking simulation results are illustrated in Figure 6. It is possible to note that after some transient period the geometric path is tracked satisfactorily.

In order to analyse the sensitivity of the control system to the variations of the gain matrix $K$ two sets of simulations are conducted. In the first set of simulations the linear error gain is set to $K_l = -0.1$ while the heading angle error gain

![Figure 5. Tested trajectories. The red line represents the desired geometric path and the blue line represents the WMR motion behavior.](image)

![Figure 6. A specific trajectory with smooth and sharp curvature sections. Selected gains are $K_l = -0.1$ and $K_{\theta} = -10.0$.](image)
$K_\theta$ is varied within the set $[-1.0, -5.0, -10.0, -35.0]$. The experimental results are illustrated in Figure 7.

In the upper part of Figure 7 the plot (a) shows the linear error $e_l$. All four curves for the linear error $e_l$ are very close which indicates that for the chosen gain $K_l = -0.1$ the linear error dynamics is decoupled from the dynamics of the heading angle error $e_\theta$. One must note that sharp curvature sections arise on time instants $t = 5\text{sec}$, $t = 15\text{sec}$, $t = 25\text{sec}$ and $t = 35\text{sec}$.

In the bottom part of Figure 7 the plot (b) illustrates the heading angle error $e_\theta$. Overall, the heading angle error $e_\theta$ remains relatively small ($<0.5\text{ rad}$) but increases with the increase of time and oscillates in the sharp curvature sections.

While increasing the module of the heading angle gain $K_\theta$, the error decreases. However, for $K_\theta = -35$ bounded high frequency chattering signals are observed.

In the second set of simulations the linear error gain is varied within the set $K_l = [-0.1, -0.2, -0.5, -1.0]$ while the heading angle error gain is set to $K_\theta = -10.0$. The experimental results are illustrated in Figure 8.

![Figure 7](image.png)

Figure 7. The (a) Linear error $e_l$ and (b) the heading angle error $e_\theta$ for $K_l = -0.1$ and $K_\theta = [-1.0, -5.0, -10.0, -35.0]$.

![Figure 8](image.png)

Figure 8. The (a) Linear error $e_l$ and (b) the heading angle error $e_\theta$ for $K_l = [-0.1, -0.2, -0.5, -1.0]$ and $K_\theta = -10.0$. 
In the upper part of Figure 8 the plot (a) shows the linear error $e_l$. It is possible to note that as the linear error gain $K_l$ increases the transient response becomes faster with asymptotic behaviour. In the bottom part of Figure 8 the plot (b) shows the heading angle error $e_\theta$. On the contrary, as the linear error gain $K_l$ increases the heading angle error $e_\theta$ increases and becomes oscillatory and even with bounded high frequency chattering signals, specially for $K_l = -0.5$ and $-1.0$.

5. Conclusions and future work

In this work, a trajectory tracking control system for the kinematics of an underactuated wheeled mobile robot has been proposed. The main rationale is related to the use of feedback linearization and the Follow the Carrot algorithm.

The Follow the Carrot algorithm is based on forcing the WMR to track the trajectory defined by the movement of a Carrot point $C_p$. The control system performance is defined by a 2-dimensional error vector, whose components are the linear error $e_l$ and the heading error $e_\theta$.

The use of feedback linearization allows the dynamics of the error vector to become linear. The controller is expressed as a $2 \times 2$ diagonal matrix whose elements in the main diagonal are the gains $K_l$ for the linear error $e_l$ and $K_\theta$ for the heading angle error $e_\theta$. The control system design is then reduced on the choice of $K_l$ and $K_\theta$.

Some experimental results have been obtained via computing simulations using the Simulink/Matlab environment. Some examples of trajectories have been defined for the experiments: a straight line and a circular pattern, both with constant velocity; a sinusoidal pattern with velocity component in the $y$ direction and a spiral trajectory with exponential velocity. It has been shown that with chosen gains $K_l = -0.1$ and $K_\theta = -2.0$ the trajectories are tracked very satisfactorily.

In order to investigate the sensitivity of the control system to variations of gains $K_l$ and $K_\theta$ a more challenging trajectory is proposed with a mix of smooth and sharp curvature sections. Two set of simulations are devised. In the first set of simulations $K_l$ remains constant while $K_\theta$ varies. It is observed that the performance of the linear error $e_l$ is robust against variations of $K_\theta$ but the performance of the heading angle error $e_\theta$ decreases with the increase of the module of $K_\theta$.

In the second set of simulations $K_\theta$ remains constant while $K_l$ varies. It is possible to note that as the linear error gain $K_l$ increases the transient response becomes faster with asymptotic behaviour. The performance of the heading angle error $e_\theta$ decreases with the increase of the linear error gain $K_l$.

In principle it seems that the linear error $e_l$ is decoupled from the heading error $e_\theta$ but the opposite is not true. A more detailed analysis is required. Therefore, future work must comprise a more detailed analytical investigation about system instability. And also the dynamic part of the WMR model must be incorporated for a more realistic analysis of the control system performance.

6. ACKNOWLEDGEMENTS

This research is being developed under the Inter institutional Doctoral Research Program, between the IFSC (Instituto Federal de Santa Catarina) and the EPUSP (Escola Politécnica da Universidade de São Paulo), sponsored by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES). Authors would like to acknowledge the support of the IFSC, the EPUSP and the CAPES.

7. REFERENCES


Robotics Institute, Pittsburgh, PA.


Encarnação, P. and Pascoal, A., 2002. “Combined trajectory tracking and path following control for dynamic wheeled mobile robots”. In IFAC World Congress.


8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.